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One Dimensional Irreversible Gas Flow in Nozzles

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ONE DIMENSIONAL IRREVERSIBLE GAS FLOW IN NOZZLES

by

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SUMMARY

Several models of one-dimensional irreversible flow of perfect gas in nozzles are discussed. Using the so-called "friction parameter" model the basic properties of irreversible nozzle flows are determined for both adiabatic and diabatic conditions. This is followed by a study of nozzle flows using concepts of flow polytropy and efficiency; here it is shown how the "small stage efficiency" may be used to explain diabatic flows.

Throughout the paper the overall objective is to give a clear but concise summary of one-dimensional nozzle flow theories and to point out their inadequacies.

ONE DIMENSIONAL IRREVERSIBLE GAS FLOW IN NOZZLES

1. Introduction

Many authors have considered the problem of one-dimensional irreversible compressible flow and in recent years several useful contributions have been made to the theory of such flows (Hicks et alia 1947; Shapiro and Hawthorne 1947; Kestin and Oppenheim 1948; Naylor 1951a, 1951b, 1952; Spalding et alia 1951, 1952; Kestin and Zaremba 1953; Mayhew and Rogers 1953; Saunders 1953; Shapiro 1953; Crocco 1958). In this paper some simple theories of irreversible gas flow in nozzles are discussed. These theories are based on assumptions of steady, shock free flow without separation and the usual considerations involved in the one-dimensional approximation of fluid motion. Within these bounds it is only possible to deal with irreversible flow in "small" nozzles, the main virtue being tractable results and a relatively neat explanation of the physics involved.

Generally speaking the theories discussed may be classified as "adequate" mathematical models, not only because it is impossible to deal properly with viscous effects in one-dimensional theory but also because average values of flow properties are used and the relationship between heat transfer and boundary layer friction is not considered in some of them.

In nozzles where there is appreciable growth of the boundary layer the average values of actual flow properties vary from point to point along the nozzle; there is no possibility of dealing with this variation and at the same time retaining a strictly one-dimensional approach. Similarly one-dimensionality implies constant temperature over any cross section, the temperature variation along the nozzle depending on both the flow and temperature distribution along the walls of the nozzle. To eliminate the influence of the wall temperature it is often assumed the walls are adiabatic boundaries. This implies no heat transfer from the surroundings and no heat conduction along the wall. The assumption is justifiable as long as one is satisfied with a rough theory; certainly it has been found that adiabatic conditions are sufficiently representative of many practical nozzle flows. We shall not limit ourselves to adiabatic flows but shall also give some consideration to the more general case of diabatic flow with heat transfer from the walls.

In particular we shall use two different approaches to the problem of formulating irreversible gas flow in nozzles. The first involves the usual hydraulic treatment of viscous effects. Viscous action is assumed localised at the walls and thus is dealt with in terms of a friction parameter or friction factor.

The second, and older, method is based on an assumption of polytropic flow. It is less exact than the friction parameter model but is widely used in practice. The approach hinges on concepts of efficiencies, those used here being the so-called "adiabatic efficiency" and "small stage efficiency". It is shown how the latter may be used to represent diabatic flows and the efficiency models are then used to study both adiabatic and diabatic constant efficiency nozzle flow.

For simplicity consideration is limited to perfect gases and nozzles of circular cross-section.

2. Notation

A	Area
$C = (\gamma R T)^{1/2}$	Acoustic velocity
C_p	Specific heat at constant pressure
C_v	Specific heat at constant volume
D	Diameter
f	Friction factor
$h = C_p T$	Specific enthalpy
k	Wall heat transfer coefficient BTU/ft^2 $^\circ$R sec.
$M = V/C$	Local Mach Number
$\dot{m} = \rho A V$	Mass flow rate
$N = V/V_{max}$	Reduced velocity
n	Polytropic state path index
p	Static pressure
p_o	Isentropic stagnation pressure
Q	Heat added per unit mass
R	Gas constant
r	Radius of curvature of generatrix at throat of nozzle
S	Specific entropy
s	Surface area of nozzle wall
T	Static Temperature
T_o	Adiabatic stagnation temperature
T_w	Wall temperature
T_{aw}	Adiabatic wall temperature
V	Velocity
$V_{max} = (2C_p T_o)^{1/2}$	Maximum velocity attainable from stagnation temperature T_o
x	Distance along nozzle axis (positive in direction of flow)
$\gamma = C_p/C_v$	Ratio of specific heats
η_{ad}	Adiabatic efficiency

H_{ad}	Overall Adiabatic nozzle efficiency
η_s	Small stage efficiency
ϕ	Wall tangent angle (see Fig. 1)
ρ	Density
τ	Wall shear stress
ξ	Recovery factor

Suffixes

t	Conditions at throat of nozzle
e	Conditions at exit
n	Interval number (see equation 5
1	Datum point in nozzle

* indicates values under critical (M = 1) conditions.

Primed[1] variables indicate values for isentropic process.

3. The Adiabatic Wall Friction Parameter Model

3.1 Equations of Steady One-dimensional Adiabatic Iso-energetic Gas Flow

Consider a perfect gas for which the equation of state is,

$$p/\rho = RT \;, \tag{1}$$

or in differential form,

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \,. \tag{2}$$

Local Mach Number is defined by

$$M^2 = \frac{V^2}{\gamma RT} \;, \tag{3}$$

from which

$$\frac{dM^2}{M^2} = \frac{dV^2}{V^2} - \frac{dT}{T} \tag{4}$$

is obtained.

The one-dimensional equations of conservation of mass and energy are, respectively,

$$\rho AV = \dot{m} \tag{5}$$

and

$$C_p T + \frac{V^2}{2} = C_p T_o \;, \tag{6}$$

which, on differentiation, give

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \tag{7}$$

and

$$\frac{dT}{T} + \frac{\gamma - 1}{2} M^2 \frac{dV^2}{V^2} = 0 \ . \qquad (8)$$

It is noted that T_o is of constant magnitude in an iso-energetic adiabatic flow : also from (6)

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2 \ , \qquad (9)$$

and it follows that

$$\frac{dT}{T} + \frac{\frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M^2} \frac{dM^2}{M^2} = 0 \ . \qquad (10)$$

Another useful flow property is the isentropic stagnation pressure, p_o, defined by

$$\frac{p_o}{p} = \frac{T_o}{T}^{\gamma/(\gamma - 1)} \qquad \text{or,}$$

$$\frac{p_o}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma - 1)} \qquad (11)$$

, by equation (9)

Note that whilst p_o is of constant magnitude in isentropic flows it is not constant in adiabatic flows. It follows from (11) that

$$\frac{dp_o}{p_o} = \frac{dp}{p} + \frac{\gamma M^2}{1 + \frac{\gamma - 1}{2} M^2} \frac{dM^2}{M^2} \qquad (12)$$

3.2 The Momentum Equation

Irreversibility due to wall friction can be included in one-dimensional theory by considering viscous effects only to manifest themselves at the walls in the form of shear stresses τ. For a nozzle of circular cross-section it is logical to assume axi-symmetric and uniform conditions at any section and hence to define the problem as shown in Figure 1.

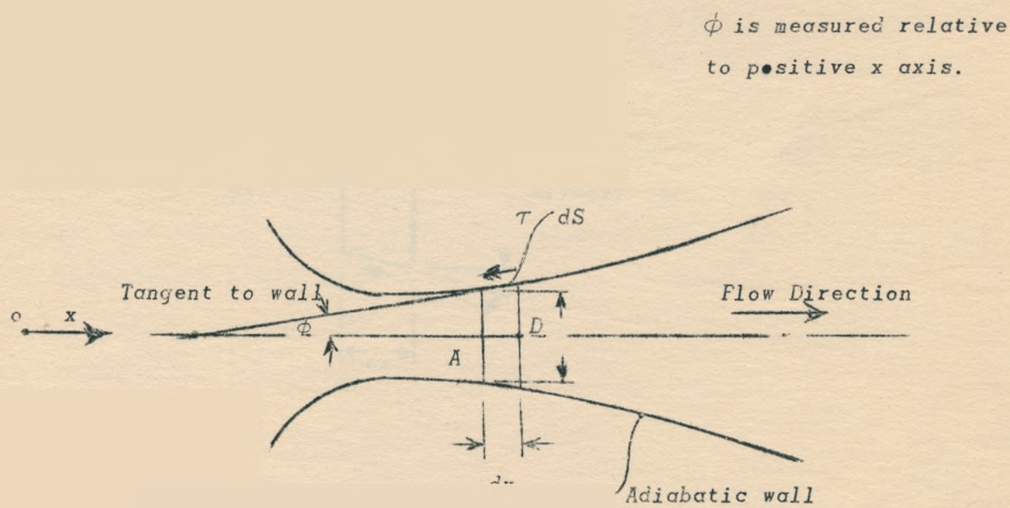

Figure 1.

Here τ is shown to be directed along the tangent to the wall. That is to say it will be directed at an angle ϕ measured positive relative to the x positive axis.

From the geometry of Figure 1 we see

$$ds = \pi D \, dx/\cos\phi$$

$$= 4A \, dx/(D\cos\phi) \qquad (13)$$

Now if we apply the momentum theorem, considering what happens to the fluid as it passes through the duct length dx we find, vide Figure 2.,

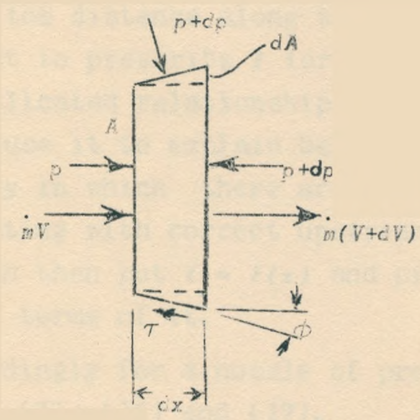

Figure 2.

$$pA - (p + dp)(A + dA) + (p + dp)dA - \tau ds\cos\phi = \dot{m}(V + dV) - \dot{m}V \, . \quad (14)$$

Simplifying this result to

$$dp + \frac{\tau ds \cos\psi}{A} + \frac{m}{A} dV = 0 \, , \quad (15)$$

and then using (5) and (13) we obtain

$$dp + \frac{4}{D}\tau dx + \rho V \, dV = 0, \quad (16)$$

where the shear stress τ can be conveniently dealt with by introducing the friction parameter law,

$$\tau = f. \, \rho V^2/2 \, . \quad (17)$$

In doing so, the use of the friction factor f gives rise to some difficulties. From knowledge of adiabatic irreversible gas flow in constant area ducts it is clear that f may depend on both the local Reynolds and Mach numbers as well as on the shape of the passage. Only for a prescribed nozzle geometry and fixed mass flow rate will it be possible to consider f varies uniquely in x, the distance along the nozzle. However, whilst it may be difficult to prescribe f for design purposes because of its rather complicated relationship to other flow properties it is feasible to use it to explain behaviour in a nozzle of prescribed geometry in which there are no shocks. Provided the nozzle is operating with correct upstream and downstream pressures we can then put $f = f(x)$ and proceed to evaluate real flow effects in terms of it.

Accordingly for a nozzle of prescribed geometry we find using (1), (3), (16) and (17),

$$\frac{dp}{p} + \frac{1}{2}\gamma M^2 \frac{dV^2}{V^2} + \frac{4f}{2D}\gamma M^2 dx = 0 \qquad (18)$$

which characterises the influence of irreversibility due to friction. Actually (18) also applies to a parallel duct and thus is compatible with the well known Fanno line process. Area change is introduced by writing

$$\left.\begin{aligned} \frac{dA}{A} &= 2\frac{dD}{D} \\ \frac{dD}{dx} &= 2\tan\phi \end{aligned}\right\} \qquad (19)$$

by which

$$\frac{4dx}{D} = \frac{1}{\tan\phi}\frac{dA}{A} \qquad (20)$$

may be substituted into (18) to give

$$\frac{dp}{p} + \frac{\gamma M^2}{2}\,\frac{dV^2}{V^2} + \frac{\gamma M^2 f}{2\tan\phi}\,\frac{dA}{A} = 0 \qquad (21).$$

Note that isentropic flow is included in (21) as a special case. By putting $f = 0$ a well known isentropic flow equation is obtained.

3.3 The Effects of Friction

The influence of friction may be discovered by using (21) in conjunction with the equations of Article 3.1.

Thus, by (10) and (4) we find,

$$\frac{dV^2}{V^2} = \frac{2dV}{V}$$

$$= \left[\frac{1}{1 + \frac{\gamma - 1}{2}M^2}\right]\frac{dM^2}{M^2} \qquad (22)$$

also by (2), (7), (10) and (22) we find

$$\frac{dp}{p} = -\frac{1}{2}\left[\frac{1 + (\gamma - 1)M^2}{1 + \frac{\gamma - 1}{2}M^2}\right]\frac{dM^2}{M^2} - \frac{dA}{A}\,. \qquad (23)$$

On putting results (22) and (23) into (21) we obtain

$$\frac{dM^2}{M^2} = 2\left[\frac{1 + \frac{\gamma - 1}{2}M^2}{M^2 - 1}\right]\left(1 - \frac{\gamma M^2 f}{2\tan\phi}\right)\frac{dA}{A}\,, \qquad (24)$$

which for our purposes is more conveniently written as

$$\frac{dM}{dx} = \frac{M}{A}\left[\frac{1 + \frac{\gamma - 1}{2}M^2}{M^2 - 1}\right]\left(1 - \frac{\gamma M^2 f}{2\tan\phi}\right)\frac{dA}{dx}\,. \qquad (25)$$

Equation (25) shows in which direction M will change depending upon the magnitude of the Mach Number, the magnitude of the term involving the friction factor and the sign of dA/dx.

Similar equations for other properties of the flow are readily established by using (24). For example, it follows from (10) and (24) that

$$\frac{dT}{dx} = -\frac{T}{A}\frac{(\gamma - 1)M^2}{M^2 - 1}\left(1 - \frac{\gamma M^2 f}{2\tan\phi}\right)\frac{dA}{dx} . \qquad (26)$$

Also by (22) and (24) we see

$$\frac{dV}{dx} = \frac{V}{A} \cdot \frac{1}{M^2 - 1}\left(1 - \frac{\gamma M^2 f}{2\tan\phi}\right)\frac{dA}{dx} \qquad (27)$$

and by (12(, (21), (22) and (27),

$$\frac{dp_o}{dx} = -\frac{p_o}{A}\left(\frac{\gamma M^2 f}{2\tan\phi}\right)\frac{dA}{dx} . \qquad (28)$$

Equations (25) to (28) are sufficient to outline the principal effects of friction as shown in Tables 1 and 2. The rather complicated singularity which exists under critical ($M = 1$) conditions is not brought out here, simply because we tacitly consider the directions of changes in properties to either side of the critical point; [detailed studies of the singularity may be found in the literature (Kestin and Zaremba 1953; Shapiro 1953).]

TABLE 1

Friction in a Convergent Duct.

(dA/dx and $\tan\phi$ are both negative)

	FLOW INITIALLY SUBSONIC $M<1$		FLOW INITIALLY SUPERSONIC $M>1$	
	I	II	III	IV
	$\frac{\gamma M^2 f}{2\tan\phi} < 1$	$\frac{\gamma M^2 f}{2\tan\phi} > 1$	$\frac{\gamma M^2 f}{2\tan\phi} < 1$	$\frac{\gamma M^2 f}{2\tan\phi} > 1$
$\frac{\gamma M^2 f}{2\tan\phi}$	—	—	—	—
$\frac{dM}{dx}$	+	+	—	—
$\frac{dV}{dx}$	+	+	—	—
$\frac{dT}{dx}$	—	—	+	+
$\frac{dp_o}{dx}$	—	—	—	—

TABLE 2

Friction in a Divergent Duct

(dA/dx and $\tan\phi$ are both positive)

	FLOW INITIALLY SUBSONIC $M<1$		FLOW INITIALLY SUPERSONIC $M>1$	
	V	VI	VII	VIII
	$\frac{\gamma M^2 f}{2\tan\phi} < 1$	$\frac{\gamma M^2 f}{2\tan\phi} > 1$	$\frac{\gamma M^2 f}{2\tan\phi} < 1$	$\frac{\gamma M^2 f}{2\tan\phi} > 1$
$\frac{\gamma M^2 f}{2\tan\phi}$	+	+	+	+
$\frac{dM}{dx}$	—	+	+	—
$\frac{dV}{dx}$	—	+	+	—
$\frac{dT}{dx}$	+	—	—	+
$\frac{dp_0}{dx}$	—	—	—	—

In summarising the effects of friction we immediately note p_0 always decreases irrespective of whether the duct is divergent or convergent and for both subsonic and supersonic initial flow conditions. On the other hand, for a convergent duct it is clear M, V, and T will vary in opposite directions for subsonic and supersonic initial conditions irrespective of the magnitude of $\gamma M^2 f/2\tan\phi$. But, as shown in Table 2, flow in a divergent duct will behave quite differently. In this case M, $?$ and T can vary in opposite directions for either subsonic or supersonic initial conditions depending on whether $\gamma M^2 f/2\tan\phi$ is less or greater than unity.

Now as far as convergent - divergent nozzles with adiabatic walls are concerned* the results shown in Tables 1 and 2 may be grouped to cover those types of flow which are physically meaningful. Firstly, from the direction of dM/dx in columns I, II, V and VI it is clear that an initially subsonic flow can be made supersonic only if sonic speed occurs downstream of the minimum area. That is to say a process determined by the conditions dictated in columns I (or II), VI and VII is necessary. We can also conclude that the Mach number at the tip of a convergent nozzle should be subsonic but that sonic conditions may occur outside the nozzle (due to vena contracta).

Three sets of conditions which basically represent meaningful operating characteristics of convergent - divergent nozzles are as follows.

Case 1 Columns I and V

Flows which comply with conditions in columns I and V will be subsonic. The factor determining this type of flow is

$$\frac{\gamma M^2 f}{2\tan\phi} < 1$$

at all points.

Case 2 Columns II, VI and VIII

Flows which comply with conditions in columns II, VI and VIII will be either entirely subsonic or subsonic with critical flow at a point downstream of the throat.

The factor determining this is

$$\frac{\gamma M^2 f}{2\tan\phi} > 1 \qquad \text{at all points.}$$

* Similar conclusions may be obtained for diffusers.

Case 3 Columns I or II, VI and VII

These conditions are compatible with the normal behaviour of supersonic nozzles and indicate that it is necessary for $\gamma M^2 f/2\tan\phi$ to become greater than unity in the subsonic portion before the throat, to remain greater than unity up to the critical point and to be less than unity in the supersonic portion of the nozzle. Thus at the critical section it is necessary that $\gamma M^2 f/2\tan\phi$ be exactly equal to one.

The situation is shown in Figure 3, where the semi-angle at the critical section ϕ^* is the minimum angle necessary for the production of supersonic velocities.

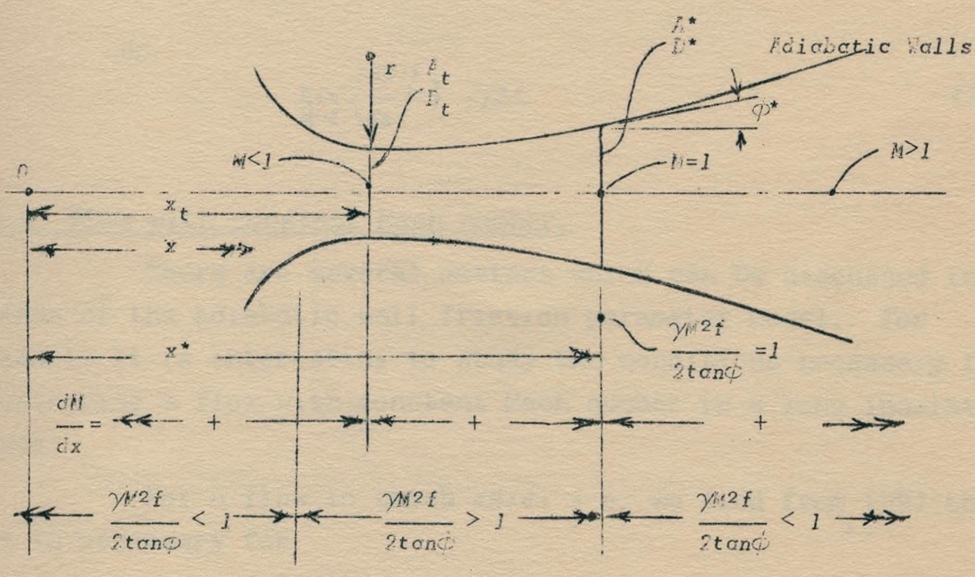

Figure 3

The magnitude of ϕ^* is found from (25) and (19), whence we have

$$\frac{dM}{dx} = \frac{2M}{D}\left[\frac{1+\frac{\gamma-1}{2}M^2}{M^2-1}\right](2\tan\phi - \gamma M^2 f). \tag{29}$$

For $M > 1$:

$$\frac{dM}{dx} > 0 .$$

it follows that

$$\tan\phi > \frac{\gamma M^2 f}{2}$$

with a limiting value for critical $(M=1)$ conditions

$$\lim_{M\to 1} \phi^* \nless \arctan \frac{\gamma}{2} f^* \tag{30}$$

or

$$\lim_{M\to 1}\left(\frac{dD}{dx}\right)^* \nless \gamma f^* \tag{31}$$

3.4 Flow with constant Mach number.

There are several matters which can be discussed in terms of the adiabatic wall friction parameter model. For example it is interesting to study the conditions necessary for sustaining a flow with constant Mach number in a long insulated duct.

For a flow in which $dM/dx = 0$, we find from (29) that it is necessary for

$$\frac{\gamma M^2 f}{2\tan\phi} \equiv 1. \tag{32}$$

In view of this (26), (27) and (32) indicate

$$\left.\begin{array}{l}\dfrac{dT}{dx}\\[2em] \dfrac{dV}{dx} = 0\ .\end{array}\right\} \tag{33}$$

Similarly, by (28) and (32),

$$\frac{dp_o}{dx} = -\frac{p_o}{A}\frac{dA}{dx} \tag{34}$$

and using (21) and (27) to obtain

$$\frac{dp}{dx} = -\frac{p}{A}\left\{\frac{\gamma M^2}{M^2-1} + \frac{\gamma M^2 f}{2\tan\phi}\left(1 - \frac{\gamma M^2 f}{2\tan\phi}\right)\right\}\frac{dA}{dx} \tag{35}$$

and then employing (32) we find,

$$\frac{dp}{dx} = -\frac{p}{A}\ \frac{dA}{dx}\ . \tag{36}$$

We conclude from (34) and (36) that

$$\left.\begin{array}{rl} & pA = \text{constant} \\ \text{and} & p_oA = \text{constant}\end{array}\right\} \tag{37}$$

are basic properties of this type of flow. As p_o always decreases, we see

$$\frac{A}{A_1} = \frac{p_{o1}}{p_o} > 1\ , \tag{38}$$

where suffix $_1$ refers to an upstream datum point and conclude that the duct must diverge whether the constant Mach number be subsonic or supersonic. Even if it is possible to have completely insulated walls it is unlikely such ducts could be designed using equation (32); mainly because the friction factor is not constant but varies with the constantly changing Reynolds number of the flow. Studies of flow in divergent tubes confirm this. (Barna, 1954, page 31)

3.5 The critical diameter D^* and its location

That critical conditions must occur downstream of the throat if the walls are adiabatic boundaries and irreversibility is solely due to wall friction has already been established. So far nothing has been done which tells just how far downstream the critical conditions would be.

Now in the region near the throat we can write (see Figure 3),

$$\frac{dD}{dx} \approx (x - x_t)\left(\frac{d^2D}{dx^2}\right)_t , \tag{39}$$

where

$$\left(\frac{d^2D}{dx^2}\right)_t = \frac{2}{r} . \tag{40}$$

It follows that

$$x - x_t \approx \frac{r}{2}\,\frac{dD}{dx} \tag{41}$$

and hence for the critical point, putting $x = x^*$,

$$x^* - x_t \approx \frac{r}{2}\left(\frac{dD}{dx}\right)^* \tag{42}$$

$$\approx \frac{r}{2}\,\gamma f^* \quad \text{by (31).} \tag{43}$$

The magnitude of the critical diameter is found by making a Taylor series expansion of the nozzle shape function $D(x)$ about the point $x = x_t$ (Crocco 1958). On doing so we obtain

$$D(x) = D_t + \left(\frac{dD}{dx}\right)_t (x - x_t) + \left(\frac{d^2D}{dx^2}\right)_t \frac{(x - x_t)^2}{2} \tag{44}$$

+ small order terms

in which,

$$\left(\frac{dD}{dx}\right)_t = 0 \qquad \text{(c.f. equation (39))}$$

and

$$\left(\frac{d^2D}{dx^2}\right)_t = \frac{2}{r} \tag{40 bis}$$

Thus

$$D(x) = D_t + \frac{1}{r}(x - x_t)^2 \tag{45}$$

and for the critical diameter $D^* = D(x^*)$

$$D^* = D_t + \frac{1}{r}(x^* - x_t)^2 . \tag{46}$$

The approximate magnitude of D^* is found by means of (43) and (46). It is

$$D^* \approx D_t + r\left(\frac{\gamma f^*}{2}\right)^2 \tag{47}$$

or in non-dimensional form,

$$\frac{D^*}{D_t} \approx 1 + \frac{r}{D_t}\left(\frac{\gamma f^*}{2}\right)^2 . \tag{48}$$

Generally, for short nozzles f^* will be about 0.01 and r/D_t about unity. That is to say for short nozzles

$$x^* - x_t \sim \frac{D_t}{(200/\gamma)} \tag{49}$$

and

$$\frac{D^*}{D_t} - 1 \sim 10^{-4} \tag{50}$$

3.6 Flow with constant friction factor

Again, providing the walls are adiabatic boundaries it is possible to compute approximately the Mach number variation for nozzles of prescribed geometry. This is conveniently done by assuming the friction factor to be constant, there being no point in being more elegant because of the basic inadequacies of the model.

To compute the variation in Mach number (29) may be used in finite difference form, viz.,

$$\Delta M_{n+1} = \frac{2M_n}{D_1 + \sum_{o}^{i=n} (2\tan\phi_i \, \Delta x_i)} \left[\frac{1 + \frac{\gamma - 1}{2} M_n^2}{M_n^2 - 1} \right] (2\tan\phi_n - \gamma M_n^2 f) \Delta x_{n+1} \tag{51}$$

with

$$x_{n+1} = x_1 + \sum_{o}^{i=n} \Delta x_i + \Delta x_{n+1} \tag{52}$$

$$M_{n+1} = M_1 + \sum_{o}^{i=n} \Delta M_i + \Delta M_{n+1} \, . \tag{53}$$

As is indicated (51) to (53) must be applied progressively to successive small forward intervals starting with known initial values $x = x_1$; $D = D_1$; $M = M_1$.

4. The Diabatic Flow Friction Parameter Model.

4.1 Equations of Steady One-dimensional Diabatic Gas Flow.

In reality the adiabatic wall is a fictitious concept. Flow in real nozzles invariably takes place with heat transfer to or from the walls which have a temperature distribution $T_w = T_w(x)$ due to the surroundings and heat conduction within the metal itself. Because inclusion of heat transfer in the one dimensional theory makes the results relatively complicated, we shall only proceed to the point where something can be said about variations in Mach number.

We note that equations (6) and (9) are not generally valid in diabatic flow: they can be applied only locally because the stagnation temperature $T_o = T_o(x)$ varies due to the heat transfer. In fact over the duct length dx, the heat transferred through the walls is given by

$$dQ = C_p dT_o$$

$$= C_p dT + \tfrac{1}{2} dV^2 \quad . \tag{54}$$

We also have by (9),

$$\frac{dT_o}{T_o} = \frac{dT}{T} + \frac{(\gamma-1)M^2/2}{1 + \frac{(\gamma-1)}{2}M^2} \frac{dM^2}{M^2} \tag{55}$$

Equations (54) and (55) together with the momentum equation, (21), and equations (1) to (5) and (7) are sufficient for our purposes .

Transposing (4) and (55) to get dT/T in each case and then equating the results we find,

$$\frac{dV^2}{V^2} = \frac{dT_o}{T_o} + \frac{1}{1 + \frac{(\gamma-1)}{2}M^2} \frac{dM^2}{M^2}$$

$$= \frac{2dV}{V} \tag{56}$$

Also by (2), (7) and (55),

$$\frac{dp}{p} = \frac{1}{2}\,\frac{dT_o}{T_o} - \frac{1}{2}\left[\frac{1 + (\gamma-1)M^2}{1 + \frac{\gamma-1}{2}M^2}\right]\frac{dM^2}{M^2} - \frac{dA}{A} \tag{57}$$

When (56) and (57) are used in (21) there results

$$\frac{dM}{dx} = \frac{M}{A}\left[\frac{1 + \frac{\gamma-1}{2}M^2}{M^2 - 1}\right]\left(1 - \frac{\gamma M^2 f}{2\tan\phi}\right)\frac{dA}{dx} - \frac{M}{T_o}\left[\frac{1 + \frac{\gamma-1}{2}M^2}{M^2 - 1}\right]\left(\frac{1 + \gamma M^2}{2}\right)\frac{dT_o}{dx} \,. \tag{58}$$

which plays a role similar to (25) in the earlier case. Note that (58) reduces to (25) when the walls are adiabatic boundaries and $dT_o/dx = 0$. The second term on the R.H.S. can be recast to bring out heating effects more clearly.

Consider the rate of heat flow through the nozzle wall surface contained within length dx, viz.,

$$\rho AV\, dQ = k(T_w - T_{aw})\frac{\pi D dx}{\cos\phi} \tag{59}$$

By (54) and (20), equation (59) can be used to put

$$dT_o = \frac{k(T_w - T_{aw})}{\rho V C_p \sin\phi}\,\frac{dA}{A}\,, \tag{60}$$

in which the Reynold Heat Transfer Analogy,

$$k = \frac{\rho V C_p f}{2} \tag{61}$$

may be introduced together with the adiabatic wall temperature relation (Shapiro 1953, p.212)

$$\frac{T_{aw}}{T} = 1 + \xi\,\frac{\gamma-1}{2}M^2 \tag{62}$$

to get

$$dT_o = \frac{f}{2\sin\phi}(T_w - T_{aw})\frac{dA}{A} , \tag{63}$$

or

$$dT_o = \frac{f}{2\sin\phi}\left\{T_w - T\left(1 + \xi\frac{\gamma-1}{2}M^2\right)\right\}\frac{dA}{A} \tag{64}$$

Thus alternative statements of (58) are

$$\frac{dM}{dx} = \frac{M}{A}\left[\frac{1+\frac{\gamma-1}{2}M^2}{M^2-1}\right]\left\{\left(1 - \frac{\gamma M^2 f}{2\tan\phi}\right) - \frac{f}{2\sin\phi}\left(\frac{1+\gamma M^2}{2}\right)\left(\frac{T_w - T_{aw}}{T_o}\right)\right\}\frac{dA}{dx} \tag{65}$$

or

$$\frac{dM}{dx} = \frac{M}{A}\left[\frac{1+\frac{\gamma-1}{2}M^2}{M^2-1}\right]\left(1 - \frac{\gamma M^2 f}{2\tan\phi}\right)\frac{dA}{dx}$$

$$-\frac{M}{4A}\left(\frac{f}{\sin\phi}\right)\left(\frac{1+\gamma M^2}{M^2-1}\right)\left\{\frac{T_w}{T} - \left[1 + \xi\frac{\gamma-1}{2}M^2\right]\right\}\frac{dA}{dx} . \tag{66}$$

In terms of the former model we can roughly consider (65) and (66) as equivalent to

$$\frac{dM}{dx} = \left(\frac{dM}{dx}\right)_{adiabatic} - \left(\frac{dM}{dx}\right)_{diabatic} ; \tag{67}$$

although this implies the same values of friction factor in both adiabatic and diabatic cases, the error is hardly significant.

Clearly real conditions will be much more complicated than we are led to believe by the adiabatic wall model and there is no possibility of obtaining tractable statements of effects similar to those in Tables 1 and 2. Derivation of equations for dT/dx, dV/dx, dp_o/dx and dp/dx is a simple matter.

Equation (65) may be used to study Mach number variation in Nozzles of prescribed geometry if written as a difference equation, viz.,

$$\Delta M_{n+1} = \frac{2M_n}{D_1 + \sum\limits_{0}^{i=n} (2\tan\phi_i \, \Delta x_i)} \left[\frac{1 + \frac{\gamma - 1}{2} M_n^2}{M_n^2 - 1} \right]$$

$$\left\{ (2\tan\phi_n - \gamma M_n^2 f) - \frac{f}{\cos\phi_n} \left(\frac{1 + \gamma M_n^2}{2} \right) \left(\frac{T_{w_n} - T_{aw_n}}{T_{o_n}} \right) \right\} . \Delta x_{n+1} \tag{68}$$

Here the friction factor is assumed constant and the wall temperature distribution must be known.

Further information on diabatic flows with friction may be found in the literature (Crocco 1958 p.221 et seq.). An experimental study of the influence of heat transfer has been made by Barna (1954).

5. Polytropic Flow Models.

5.1 The Polytropic Flow Models.

The method of dealing with nozzle flows by assuming the real process to be polytropic dates back to the last century (Zeuner 1907). It involves an assumption of one dimensionality and the hope that all sources of irreversibility (and variations from assumed one dimensional conditions) will manifest themselves in a simple polytropic state path law. In the light of Article 4.1 and the following we shall see the method requires comparison of rather complicated and intractible real conditions and corresponding isentropic conditions between the same pressure limits. It can hardly be expected that patching of a single polytrope to the real process will be a simple matter; nevertheless these models have been found adequate and sufficient for a number of practical applications.

5.2 The Adiabatic Efficiency Model.

To introduce the idea of adiabatic efficiency it is necessary to consider the walls are adiabatic boundaries.

Suppose p_1, T_1, M_1 refer to some datum point in the subsonic portion of the nozzle. Subsequent states in the flow are assumed to follow the adiabatic state path shown in Figure 4. Equations (6) or (9) show states downstream of the datum point can be related to stagnation conditions $T_{o_1} = T_o$; $M = 0$.

Hence the adiabatic efficiency is defined as,

$$\eta_{ad} = \frac{V^2}{V'^2} = \frac{AB}{AC} \qquad (69)$$

or, by (6),

$$\eta_{ad} = \frac{T_{o_1} - T}{T_{o_1} - T'} .$$

I.e.,

$$\eta_{ad} = \frac{(1 - T/T_{o_1})}{(1 - T'/T_{o_1})} \tag{70}$$

where T'/T_{o_1} represents the temperature ratio corresponding to isentropic expansion from pressure p_{o_1} to p.

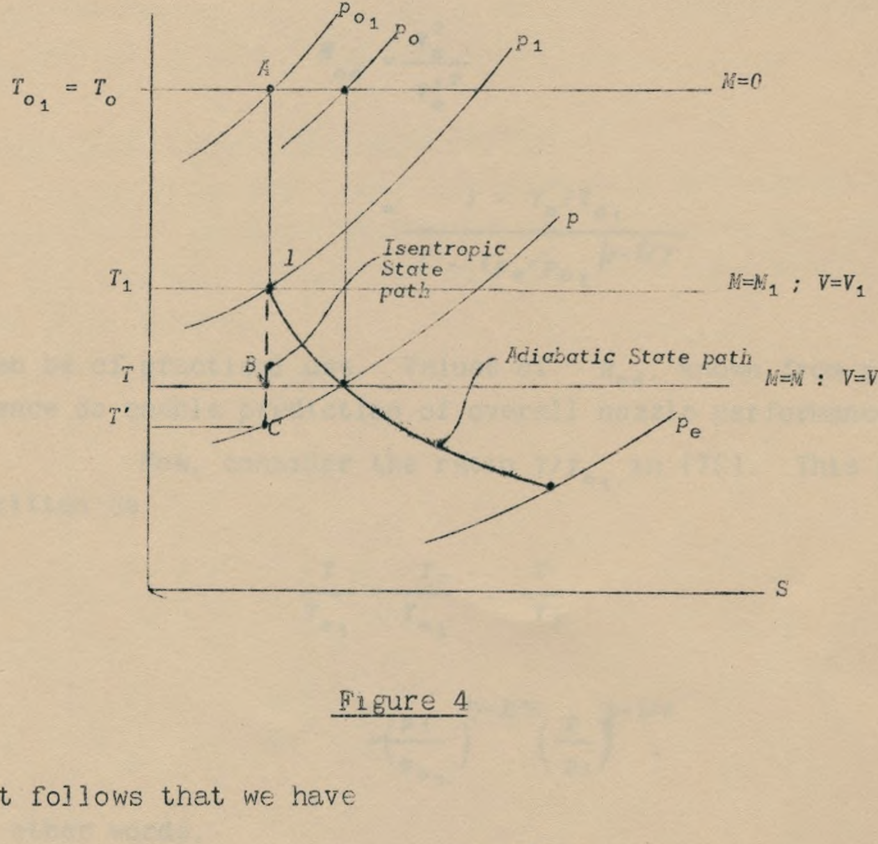

Figure 4

It follows that we have

$$\frac{T'}{T_{o_1}} = \left(\frac{p}{p_{o_1}}\right)^{\gamma-1/\gamma} \tag{71}$$

and

$$\eta_{ad} = \frac{(1 - T/T_{o_1})}{1 - (p/p_{o_1})^{\gamma-1/\gamma}} \tag{72}$$

Clearly then $\eta_{ad} = \eta_{ad}(x)$ will be a single valued function, self determined by the temperature and pressure distributions $p = p(x)$ and $T = T(x)$ of the real (adiabatic) flow conditions. It can be of use in explaining the real flow properties only if it can be pre-determined : and it cannot be! However, the overall adiabatic efficiency defined as

$$H_{ad} = \frac{V_e^2}{V_e'^2}$$

$$= \frac{1 - T_e/T_{o_1}}{1 - (p_e/p_{o_1})^{(\gamma-1)/\gamma}} \tag{73}$$

can be of practical use. Values of H_{ad}, known from past experience do enable prediction of overall nozzle performance.

Now, consider the ratio T/T_{o_1} in (72). This can be written as

$$\frac{T}{T_{o_1}} = \frac{T_1}{T_{o_1}} \cdot \frac{T}{T_1}$$

$$= \left(\frac{p_1}{p_{o_1}}\right)^{(\gamma-1)/\gamma} \left(\frac{p}{p_1}\right)^{(n-1)/n} . \tag{74}$$

In other words,

$$\eta_{ad} = \frac{1 - (p_1/p_{o_1})^{(\gamma-1)/\gamma} (p/p_1)^{(n-1)/n}}{1 - (p/p_{o_1})^{(\gamma-1)/\gamma}} \tag{75}$$

and if it is possible to represent the real process by a simple polytropic state path (n = constant) then η_{ad} cannot be of constant magnitude. Conversely should η_{ad} be of constant magnitude the process will not be a simple polytrope. The practice of specifying values of $\bar{H}_{ad}$ without making any reference to pressure limits is commonly met. On writing

$$\bar{H}_{ad} = \frac{1 - (p_1/p_{o_1})^{(\gamma-1)/\gamma} \, (p_e/p_1)^{(n-1)/n}}{1 - (p_e/p_{o_1})^{(\gamma-1)/\gamma}} \qquad (76)$$

and remembering p_1, p_{o_1}, p_e and n are all of arbitrary magnitude such practice is seen to be dangerous.

5.3 The Small-Stage efficiency model.

An alternative approach which does not require the restriction of adiabatic boundaries and which does not involve the hope that the real process can be represented by a simple polytrope is as follows.

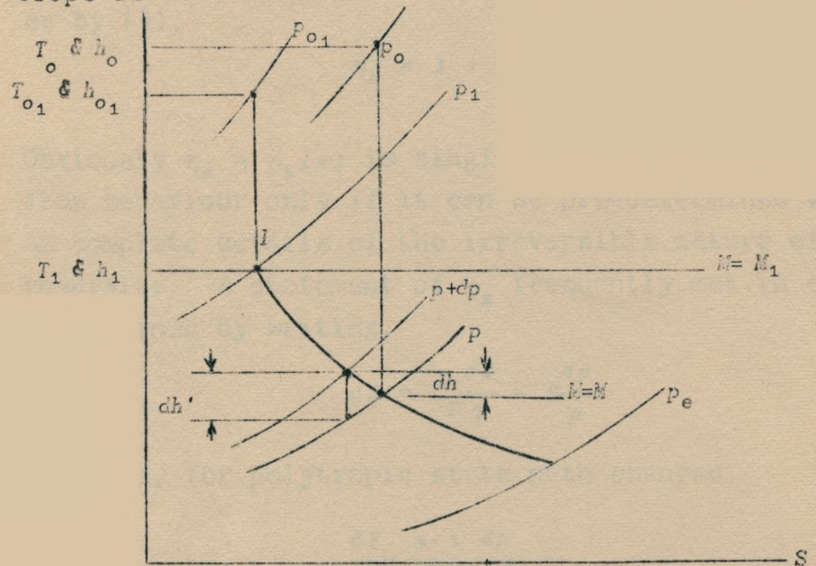

Figure 5

Consider the enthalpy change dh which occurs due to both friction and heat transfer when the pressure falls by dp over an elemental length dx of the nozzle.

By comparing this change with the corresponding isentropic enthalpy change for the same small pressure drop we define the small stage efficiency

$$\eta_s = \frac{dh}{dx} \Big/ \frac{dh'}{dx} \qquad (77)$$

Here, by the First and Second Laws of Thermodynamics

$$dh = Tds + dp/\rho \qquad (78)$$

and we find,

$$\eta_s = 1 + \rho T\left(\frac{dS}{dx}\right)\left(\frac{dx}{dp}\right) \qquad (79)$$

or by (1),

$$\eta_s = 1 + \frac{p}{R}\left(\frac{dS}{dx}\right)\left(\frac{dx}{dp}\right) \qquad (80)$$

Obviously $\eta_s = \eta_s(x)$ is single valued; it can be used to explain flow behaviour only if it can be pre-determined which is not easy as complete details of the irreversible nature of the flow is prerequisite. A statement of η_s frequently met in elementary texts is obtained by writing,

$$dS = C_p\frac{dT}{T} - R\frac{dp}{p} \qquad (81)$$

in which, for polytropic state path changes,

$$\frac{dT}{T} = \frac{n-1}{n}\frac{dp}{p} \qquad (82)$$

By (80), (81) and (82) it is found that

$$\eta_s = \frac{(n-1/n}{(\gamma-1/\gamma} \qquad (83)$$

wherein $n = n(x)$ is the index of the polytropic process in the vicinity of the point being considered. Note, how constant small stage efficiency implies a constant polytropic index and hence a simple polytropic state path.

5.4 Small stage efficiency for adiabatic flow.

The concept of small stage efficiency may be used for adiabatic flow as well as diabatic flow in nozzles and in this case it is possible to determine some of its properties.

By using (35) in (80) and remembering (c.f. equation 19)

$$\frac{dA}{dx} = \frac{4A}{D} \tan\phi \tag{84}$$

we can show

$$\eta_s = 1 - \frac{D}{4R\tan\phi\left\{\dfrac{\gamma M^2}{M^2-1} + \dfrac{\gamma M^2 f}{2\tan\phi}\left(1 - \dfrac{\gamma M^2 f}{2\tan\phi}\right)\right\}} \cdot \frac{dS}{dx} ; \tag{85}$$

hence, on considering the signs and magnitudes (see Article 3.3) of the terms in the denominator we find,

for subsonic flow upstream of the throat, $\eta_s < 1$;

for subsonic flow between the throat and the critical area where $\dfrac{\gamma M^2 f}{2\tan\phi} > +1$, $\eta_s > 1$;

and

for the supersonic flow downstream of the critical area, $\eta_s < 1$.

The unusual behaviour of η_s just downstream of the throat is not generally realised. Of course it is a product of the adiabatic wall model and is hardly meaningful when we consider real conditions.

6. Constant Efficiency Flows

6.1 The Reduced Velocity.

In conclusion let us consider some properties of a class of hypothetical flows in which the efficiencies are supposed constant. Constant efficiency flows have been studied by many authors (e.g. Naylor 1951a, 1951b, 1952; Crocco 1958) simply because it is impossible to develop the polytropic flow models any further without determining the efficiency functions which will transform isentropic conditions to corresponding real flow conditions.

There is no chance of determining these functions; and the classical method of not facing this fact and not accepting the futility of the efficiency models is to disregard the fact altogether. Thus assumption of constant efficiency is necessary if we are to squeeze the most out of an already anaemic situation.

Yet, despite the fact the flows so prescribed have no true physical counterparts, the results obtained have been found of practical use. Of course there are many other situations where a theory which is sufficiently representative of the real situation is accepted as an adequate model and there is no sense in belabouring the point.

To study constant efficiency flows it is convenient to use the dimensionless "Reduced Velocity", rather than the local Mach number.

Reduced velocity is defined as

$$N = \frac{V}{V_{max}} \quad , \tag{86}$$

where

$$V_{max} = \left(2h_o\right)^{1/2}$$

$$= \left(2C_p T_o\right)^{1/2} . \tag{87}$$

Then by (6) we have

$$\frac{T}{T_o} + \frac{V^2}{2C_p T} = 1 \ , \tag{88}$$

or

$$\frac{T_o}{T} = \frac{1}{1-N^2} \tag{89}$$

and from (9) the equivalences,

$$\frac{1}{1-N^2} = 1 + \frac{\gamma-1}{2} M^2 \tag{90}$$

$$M^2 = \frac{2}{\gamma-1} \frac{N^2}{1-N^2} \ . \tag{91}$$

6.2 Flow with constant adiabatic efficiency

Consider flow through a nozzle having adiabatic walls and assume η_{ad} is constant. The process taking place will not be a simple polytrope and because $n = n(x)$ varies in some unknown manner we do not wish to use it in the equations.

We can use (89) in (72) to find

$$\frac{p}{p_{o_1}} = \left(1 - \frac{N^2}{\eta_{ad}}\right)^{\gamma/(\gamma-1)} \tag{92}$$

where p_{o_1} is the isentropic stagnation pressure corresponding to datum state p_1, M_1 at some point in the nozzle.

Next consider the well known equation for conservation of mass in adiabatic flows,

$$\frac{\dot{m}}{A} = \left(\frac{\gamma}{RT_{o_1}}\right)^{1/2} p \ M \left(1 + \frac{\gamma-1}{2} M^2\right)^{1/2} ; \tag{93}$$

restating it in terms of reduced velocity and remembering $T_{o_1} = T_o$ is constant throughout we get

$$\dot{m}\left(RT_{o_1}\frac{\gamma-1}{2\gamma}\right)^{1/2} = \frac{pAN}{1-N^2} \qquad (94)$$

in which the L.H.S. is a constant for any given flow and the R.H.S gives the pressure - area - reduced velocity relationship for adiabatic flow.

Now if (92) is used to eliminate p in (94) we see

$$\frac{\dot{m}\left(RT_{o_1}\frac{\gamma-1}{2\gamma}\right)^{1/2}}{p_{o_1}} = \frac{AN}{1-N^2}\left(1-\frac{N^2}{\eta_{ad}}\right)^{\gamma/(\gamma-1)} . \qquad (95)$$

Here again the L.H.S. is a constant for any given flow but the R.H.S. now plays the role of an adiabatic efficiency - area - reduced velocity relationship. With η_{ad} of constant magnitude (95) becomes a simple area - reduced velocity law, by which

$$\frac{A}{A_1} = \left(\frac{N_1}{N}\,\frac{1-N^2}{1-N_1^2}\right)\left(\frac{\eta_{ad}-N_1^2}{\eta_{ad}-N^2}\right)^{\gamma/(\gamma-1)} . \qquad (96)$$

The conditions at the throat may be examined by differentiating (95). Thus

$$\frac{dA}{dx} + \frac{A}{N}\left\{\frac{1+N^2}{1-N^2} - \frac{2\gamma}{\gamma-1}\,\frac{N^2}{\eta_{ad}-N^2}\right\}\frac{dN}{dx} = 0 \qquad (97)$$

and with $dA/dx = 0$ and η_{ad} = constant the reduced velocity at the throat is found from the real root of

$$\frac{\gamma+1}{\gamma-1}N_t^4 + \left(\eta_{ad} - \frac{\gamma+1}{\gamma-1}\right)N_t^2 + \eta_{ad} = 0 \qquad (98)$$

It follows from (98) that,

$$N_t < N^* \qquad (99)$$

where

$$N^* = \frac{C^*}{V_{max}} = \left(\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{2}}, \qquad (100)$$

and hence the throat velocity is subsonic.

If we apply the model to flow with constant reduced velocity (and hence Mach number) we find it is quite inadequate. Although (94) indicates

$$pA = constant, \qquad (37\ bis)$$

equation (95) gives a silly result,

$$A = constant \qquad (101)$$

which does not comply with physical fact. The first condition is, of course, obtained because (94) is valid for any adiabatic flow. We conclude that η_{ad} cannot be constant in this case and consequently there may be other situations where the assumption can lead to equally fallacious predictions.

6.3 Diabatic flow with constant small stage efficiency

For diabatic flows in which the stagnation temperature varies (94) cannot be used with generality. However it is possible to use this equation as a local statement of conservation of mass providing T_{o_1} is replaced by T_o, the local stagnation temperature at the point considered. Hence we write,

$$\dot{m}\ (RT_o\ \{\gamma-1\}/2\gamma)^{\frac{1}{2}} = \frac{pAN}{1-N^2} \qquad (102)$$

and then recast it as,

$$\dot{m}\ (RT_{o_1}\{\gamma-1\}/2\gamma)^{\frac{1}{2}} = (T_{o_1}/T_o)^{\frac{1}{2}}\ \frac{pAN}{1-N^2} \qquad (103)$$

in order to refer conditions back to datum state (T_1, p_1, M_1).

Now, if η_s is considered to be constant throughout, the polytropic state path law exponent n must also be of constant magnitude : we then have,

$$\frac{p}{p_1} = \left(\frac{p}{p_{o_1}}\right)\left(\frac{p_{o_1}}{p_1}\right) = \left(\frac{T}{T_1}\right)^{n/n-1} , \qquad (104)$$

wherein

$$\frac{p_{o_1}}{p_1} = \left(\frac{1}{1 - N_1^2}\right)^{\gamma/\gamma-1} . \qquad (105)$$

Thus stating

$$\frac{p}{p_{o_1}} = \frac{p_1}{p_{o_1}}\left(\frac{T}{T_1}\right)^{n/n-1}$$

$$= (1 - N_1^2)^{\gamma/(\gamma-1)} (T/T_1)^{n/(n-1)} \qquad (106)$$

it is possible to eliminate p in (103) and find

$$\frac{\dot{m}(RT_{o_1} \{\gamma-1\}/2\gamma)^{\frac{1}{2}}}{p_{o_1}(1 - N_1^2)^{\gamma/\gamma-1}} = \left(\frac{T_{o_1}}{T_o}\right)^{\frac{1}{2}}\left(\frac{T}{T_1}\right)^{n/n-1} \frac{AN}{1 - N^2} \qquad (107)$$

or

$$\frac{\dot{m}(RT_{o_1} \{\gamma-1\}/2\gamma)^{\frac{1}{2}}}{p_1} = \left(\frac{T_{o_1}}{T_o}\right)^{\frac{1}{2}}\left(\frac{T}{T_1}\right)^{n/n-1} \frac{AN}{1 - N^2} \qquad (108)$$

Next write

$$\frac{T}{T_1} = \frac{T}{T_o} \cdot \frac{T_o}{T_{o_1}} \cdot \frac{T_{o_1}}{T_1}$$

$$= (1 - N^2)\frac{T_o}{T_{o_1}}\left(\frac{1}{1 - N_1^2}\right) , \qquad (109)$$

to obtain in (107) or (108)

$$\left(\frac{T}{T_1}\right)^{n/n-1} = \left(\frac{1 - N^2}{1 - N_1^2}\right)^{n/n-1} \left(\frac{T_o}{T_{o_1}}\right)^{n/n-1}. \qquad (110)$$

Hence we see from (107) that

$$\frac{\dot{m}(R\,\{\gamma-1\}/2\gamma\,)^{\frac{1}{2}}\,(1 - N_1^2)^{n/n-1}\,T_{o_1}^{n/n-1}}{(1 - N_1^2)^{\gamma/\gamma-1}\,p_{o_1}} = T_o^{\frac{n+1}{2(n-1)}}\,AN(1 - N^2)^{1/n-1} \qquad (111)$$

or

$$\frac{\dot{m}(R\,\{\gamma-1\}/2\gamma\,)^{\frac{1}{2}}\,T_1^{n/n-1}}{p_1} = T_o^{\frac{n+1}{2(n-1)}}\,AN(1 - N^2)^{1/n-1}. \qquad (112)$$

where the terms on the L.H.S. define the flow and the term on the R.H.S. gives the Stagnation temperature - area - reduced velocity relation for constant small stage efficiency. Clearly for given initial flow conditions and mass flow rate we can write,

$$\frac{A}{A_1} = \frac{N_1}{N}\left(\frac{1 - N_1^2}{1 - N^2}\right)^{1/n-1}\left(\frac{T_{o_1}}{T_o}\right)^{\frac{n+1}{2(n-1)}}. \qquad (113)$$

If the reduced velocity (i.e. Mach number c.f. 91) is constant in this type of flow we see it is necessary for

$$T_o^{\frac{n+1}{2(n-1)}}\,A = constant \qquad (114)$$

and

$$T^{\frac{n+1}{2(n-1)}}A = constant. \qquad (115)$$

Also, by differentiating (111), (112) or (113), we note

$$\frac{dA}{dx} + \frac{n+1}{2(n-1)}\frac{A}{T_o}\frac{dT_o}{dx} + \frac{A}{N}\left[1 + \frac{2N^2}{(N^2 - 1)(n-1)}\right]\frac{dN}{dx} = 0. \qquad (116)$$

and it follows from (63) that

$$\frac{dA}{dx}\left[1+\frac{n+1}{2(n-1)}\,\frac{f}{2\sin\phi}\,\frac{T_w-T_{aw}}{T_o}\right]+\frac{A}{N}\left\{1+\frac{2}{n-1}\,\frac{N^2}{N^2-1}\right\}\frac{dN}{dx}=0 \qquad (117)$$

Thus at the throat where $dA/dx = 0$ it is seen heat transfer plays no role in determining the local conditions. The throat velocity is determined solely by the term within the curly brackets; it is

$$N_t=\left(\frac{n-1}{n+1}\right)^{1/2}$$

$$=\left(\frac{\eta_s}{\{2\gamma/(\gamma-1)\}-\eta_s}\right)^{1/2}. \qquad (118)$$

Another statement of (118) using local Mach number is

$$M_t=\left(\frac{n-1}{\gamma-1}\right)^{1/2} \qquad (119)$$

and it is seen from either (118) or (119) that

$$\lim_{\eta_s\to 1} N_t=\left(\frac{\gamma-1}{\gamma+1}\right)^{1/2}=N^*\ ; \qquad (120)$$

$$\lim_{\eta_s\to 1} M_t = 1. \qquad (121)$$

That is to say the flow at the throat will always be subsonic.

The influence of heating or cooling on the reduced velocity becomes apparent on restating (117) in the form

$$\frac{dN}{dx}=-\frac{N}{A}\left[\frac{1+\dfrac{n+1}{2(n-1)}\,\dfrac{f}{2\sin\phi}\,\dfrac{T_w-T_{aw}}{T_o}}{1+\dfrac{2N^2}{(1-n)(1-N^2)}}\right]\frac{dA}{dx}\ . \qquad (122)$$

Upstream of the throat $N^2 < (n-1)/(n+1)$ and the denominator is positive ; dA/dx is negative and therefore the sign of dN/dx is determined by the term

$$1 + \frac{n+1}{2(n-1)} \frac{f}{2\sin\phi} \frac{T_w - T_{aw}}{T_o} \quad . \qquad (123)$$

With heat addition $(T_w > T_{aw})$ the reduced velocity must increase. The influence of heat extraction depends on whether

$$\frac{T_w - T_{aw}}{T_o} \gtrless \frac{2(n-1)}{(n+1)} \frac{2\sin\phi}{f} \quad . \qquad (124)$$

If cooling is predominant the flow is retarded but if friction predominates the behaviour is essentially the same as with heat addition. It can be anticipated from the likely magnitudes of the variables that friction will generally be predominant except, possibly, near the throat.

Downstream of the throat where $N^2 > (n-1)/(n+1)$ and dA/dx is positive we find heat addition causes N to increase and cooling a decrease. However the influence of cooling is still dependent on (124), the statement above also applying here, except that friction now plays a much stronger role.

6.4 Adiabatic flow with constant small stage efficiency

For adiabatic flow (94) is applicable and using (106) and (110), with $T_o = T_{o_1}$, there results

$$\frac{\dot{m}(RT_{o_1}\{\gamma-1\}/2\gamma\,)^{1/2}}{p_{o_1}(1 - N_1^2)^{\left\{\frac{\gamma}{\gamma-1} - \frac{n}{n-1}\right\}}} = AN(1 - N^2)^{1/n-1} \qquad (125)$$

or

$$\frac{\dot{m}(RT_{o_1}\{\gamma-1\}/2\gamma\,)^{1/2}(1 - N_1^2)^{n/n-1}}{p_1} = AN(1 - N^2)^{1/n-1} \qquad (126)$$

in which terms on the L.H.S. refer to the datum state. Thus by (125), (126) or (113) we find

$$\frac{A}{A_1} = \frac{N_1}{N}\left(\frac{1 - N_1^2}{1 - N^2}\right)^{1/(n-1)} . \qquad (127)$$

By differentiating or using (116) it is found that

$$\frac{dA}{dx} + \frac{A}{N}\left[1 + \frac{2N^2}{(N^2 - 1)(n-1)}\right]\frac{dN}{dx} = 0 \qquad (128)$$

which leads us to conclude

$$N_t = \left(\frac{n-1}{n+1}\right)^{1/2} . \qquad (118 \text{ bis})$$

and

$$dN/dx > 0 \text{ at all points.} \qquad (129)$$

Although (94) shows it is necessary for pA to be of constant magnitude if the Reduced velocity is constant we find (125) and (126) indicate $A = constant$. Thus we conclude η_s cannot be of constant magnitude in this type of flow.

PREVIOUS ISSUES